Syllabus template

Semester: 8	
Programme: Mathematics	
Course: Measure and Integration	
Paper Code:C4MT230821T	**Credits: 6**
Hours/week: 4	
Category: Core/MDC/SEC/VAC: Core (Elective)	
Theory / Practical / composite: Theory	
No of Modules: Nil	

Course Overview: Measure and Integration

This course develops the Lebesgue theory of measure and integration, rigorously replacing the classical Riemann framework of integration. It is structured in three progressively linked units — Lebesgue measure, measurable functions, and the Lebesgue integral. The first unit constructs Lebesgue outer measure, establishing its monotonicity, translation invariance, countable subadditivity, and agreement with length on intervals. The Cantor ternary set illustrates the theory's subtlety — an uncountable set of measure zero. Lebesgue measurable sets are then identified via countable additivity and shown to form a σ-algebra containing all

intervals, open, closed, and Borel sets, though not every subset of R is measurable. The second

unit transfers measurability to functions, with approximation of measurable functions by simple functions as the central technique. Key results include Egoroff's and Lusin's theorems — the rigorous form of Littlewood's principles — along with convergence in measure, almost uniform convergence, and the F. Riesz theorem. The third unit builds the Lebesgue integral, motivated by the shortcomings of the Riemann integral and shown to be its genuine generalization. Its crowning results — Fatou's Lemma, the Monotone Convergence Theorem, and the Dominated Convergence Theorem — justify interchanging limits and integrals, something the Riemann theory cannot do in comparable generality. The theory developed here forms the indispensable foundation for probability, functional analysis, and modern applications of mathematical analysis.

Course Outcome: On successful completion of the course a student will be able to do the following:

1.Define and explain the concept of set functions and the intuitive notion of measure; state the fundamental properties of the Lebesgue outer measure — monotonicity, translation invariance, and countable subadditivity — and demonstrate that the outer measure of an interval equals its length, with the Cantor ternary set as a key illustration.

2. Apply the Carathéodory criterion to verify the measurability of sets in R, and prove that the collection of Lebesgue measurable sets forms a σ-algebra closed under complements, countable unions, and countable intersections, on which the outer measure is countably additive.

3. Analyze the structure of measurable sets by establishing that intervals, open sets, closed sets, and Borel sets are Lebesgue measurable, constructing equivalent characterizations of measurability in terms of open, closed, G_δ, and $F_\sigma$sets, and evaluating the existence of non-measurable subsets of $R$and sets of measure zero.

4. Construct approximations of measurable functions by sequences of simple functions and apply the theorems of Egoroff and Lusin to interpret measurable functions as "nearly continuous" and pointwise convergence as "nearly uniform," in the spirit of Littlewood's principles.

5. Compare and distinguish the modes of convergence — pointwise, almost everywhere, almost uniform, and convergence in measure — and prove the F. Riesz theorem on extraction of almost-everywhere convergent subsequences.

6. Critically evaluate the shortcomings of the Riemann integral, develop the Lebesgue integral from the bounded case to the general case, justify it as a genuine generalization of the Riemann integral using Lebesgue's criterion for Riemann integrability, and apply the Bounded Convergence Theorem, Fatou's Lemma, the Monotone Convergence Theorem, and the Lebesgue Dominated Convergence Theorem to interchange limits and integrals rigorously.

Prerequisites: Metric spaces and advanced calculus; ϵ - δanalysis of real functions and Riemann integration; sequences and series of functions (uniform convergence); elementary set theory including countable/uncountable sets and the cardinality of the continuum.

SYLLABUS

UNIT/Module	CONTENT	NO OF CLASSES	CO Mapping	COGNITIVE LEVEL
I.	**Lebesgue measure**: Set functions, Intuitive idea of measure. Lebesgue outer measure and its properties: monotonicity, translation invariance, countable subadditivity, Lebesgue outer measure of an	**18**	**CO1, CO2, CO3**	**K1, K2, K3, K4**

	interval is its length. Cantor ternary set.σ-algebras. Lebesgue measure. Lebesgue measurable sets: motivation, Properties: countable additivity of outer measure on Lebesgue measurable sets, not every subset of real numbers is Lebesgue measurable. Algebra of measurable sets: complement, countable union and intersection of measurable sets is measurable. Sets of measure zero. Intervals, open sets and closed sets of real numbers are Lebesgue measurable. Borel sets. Equivalent formulation of measurable sets in terms of open, closed,F_σ and G_δ sets.			
II.	**Measurable Functions** Properties. Approximation of a measurable function by a sequence of simple functions. Measurable functions as nearly continuous functions, Egoroff Theorem, Lusin Theorem, Convergence in Measure and F. Riesz Theorem. Almost uniform convergence.	**16**	**CO4, CO5**	**K3, K4, K5**
III.	**Lebesgue Integral**: Shortcoming of Riemann integral, Lebesgue integral of a bounded function over a set of finite measure and its properties. Lebesgue integral as a generalization of Riemann integral. Bounded Convergence Theorem, Lebesgue's criteria for Riemann integrability, Fatou Lemma, Monotone Convergence Theorem.	**18**	**CO6**	**K5, K6**

	General Lebesgue integral, Lebesgue Convergence Theorem.			

Text Books

- Measure Theory and Integration by G. de Barra.
- A Course in Lebesgue Measure and Integration by S. Kumaresan.
- H.L.Royden: Real Analysis.

Suggested readings

- Measure Theory by Donald L. Cohn.
- Measure, Integration & Real Analysis by Sheldon Axler.
- Walter Rudin: Principles of Mathematical Analysis.

Web Resources and eLearning Sources: Bazett playlist + Hunter notes + Royden ch. 2, Axler (open PDF) + Tricki strategy guides, NPTEL Measure Theory + Tao + MIT 18.125.

Evaluation: Theory CIA: **20+5+5=30** Semester Exam: **70**

Paper Structure for Theory Semester Exam Module: 7 questions each of **10** marks out of a set of **13** questions.

Course outcomes (COs) and Cognitive Level Mapping

COs	CO Description	Cognitive levels
CO1	**Define and explain** the concept of set functions and the intuitive notion of measure; state the fundamental properties of the Lebesgue outer measure — monotonicity, translation invariance, and countable subadditivity — and demonstrate that the outer measure of an interval equals its length, with the Cantor ternary set as a key illustration.	**K1, K2**
CO2	**Apply** the Carathéodory criterion to verify the measurability of sets in R, and prove that the collection of Lebesgue measurable sets forms a σ-algebra closed under complements, countable unions, and countable intersections, on which the outer measure is countably additive.	**K3, K4**
CO3	**Analyze** the structure of measurable sets by establishing that intervals, open sets, closed sets, and Borel sets are Lebesgue measurable, constructing equivalent characterizations of measurability in terms	**K3, K4, K5**

	of open, closed, G_δ, and $F_\sigma$sets, and evaluating the existence of non-measurable subsets of $R$and sets of measure zero.	
CO4	**Construct** approximations of measurable functions by sequences of simple functions and apply the theorems of Egoroff and Lusin to interpret measurable functions as "nearly continuous" and pointwise convergence as "nearly uniform," in the spirit of Littlewood's principles.	**K3, K4**
CO5	**Compare and distinguish** the modes of convergence — pointwise, almost everywhere, almost uniform, and convergence in measure — and prove the F. Riesz theorem on extraction of almost-everywhere convergent subsequences.	**K4, K5**
CO6	**Critically** evaluate the shortcomings of the Riemann integral, develop the Lebesgue integral from the bounded case to the general case, justify it as a genuine generalization of the Riemann integral using Lebesgue's criterion for Riemann integrability, and apply the Bounded Convergence Theorem, Fatou's Lemma, the Monotone Convergence Theorem, and the Lebesgue Dominated Convergence Theorem to interchange limits and integrals rigorously.	**K5, K6**